The importance of electron interaction to the negative magnetoresistance of metallic n-GaAs close to the metal - insulator transition

1997 J. Phys.: Condens. Matter 9 8657

(http://iopscience.iop.org/0953-8984/9/41/012)

J. Phys.: Condens. Matter **9** (1997) 8657–8668. Printed in the UK PII: S0953-8984(97)83339-5

The importance of electron interaction to the negative magnetoresistance of metallic n-GaAs close to the metal–insulator transition

J M Monsterleet, B Capoen and G Biskupski

Laboratoire de Spectroscopie Hertzienne (Unité de Recherche associée au CNRS 249), Université de Lille I, 59655 Villeneuve d'Ascq Cédex, France

Received 11 April 1997, in final form 4 July 1997

Abstract. Magnetoconductivity measurements on n-GaAs, doped close to the metal–insulator transition (MIT), are interpreted by a weak localization model including electron–electron interaction corrections. Measurements are analysed in the temperature range 0.3–18 K and for magnetic fields up to 1.5 T.

Analysis of the data shows that current theories elaborated for high values of the disorder parameter $k_F l_0$ continue to be applicable in the immediate vicinity of the MIT over a wide range of temperature and magnetic field. The inelastic scattering time is found to vary like T^{-1}.

Nevertheless, it has been necessary to study the relative magnitude of the different contributions to electron interactions, which are shown to partially explain the surprising values of the interaction constants.

1. Introduction

Negative magnetoresistance (NMR) has been widely observed in doped semiconductors (Fritzsche and Lark-Horovitz 1955, Morita *et al* 1984, Ootuka *et al* 1987) and is regarded as a typical feature of the impurity band conduction near a mobility edge.

Recently, great progress has been made in the understanding of the NMR of disordered systems with metallic conductivity: it is now clearly established that both the weak-localization and the electron–electron interaction effects play important parts in the conduction process. Furthermore, the study of NMR remains the only way to measure the parameters describing the various microscopic processes, such as inelastic or spin–orbit scattering.

Experimentally, Dynes *et al* (1983) and then Morita *et al* (1984) were among the first to analyse NMR in metallic materials using the weak-localization theory, but they met with difficulties in fitting their data in the whole range of magnetic field without taking into account the interaction effects. It is now well understood that these interactions contribute to diminish the positive magnetoconductance. However, the relative magnitude of the different theoretical expressions and the conditions of their application are the subject of conflicting opinions. For instance, it has been shown recently that the Zeeman splitting interaction is the dominant process in Si:P and Si:B (Bogdanovich *et al* 1995).

In a previous paper (Capoen *et al* 1993) we have presented results on n-GaAs where data have been analysed using the model of Kawabata (1980). This model takes into account the weak-localization process alone. The inelastic scattering time τ_ε of our sample has been found to vary like the inverse of temperature, in accordance with the predictions of Isawa

(1984). Nevertheless, we have shown that the magnetoconductance of Kawabata gave a poor fit for moderate magnetic fields at low temperature ($T \leqslant 1.8$ K). In order to resolve the discrepancies which appear when only weak localization is considered at high magnetic field and low temperature, we investigate in the present work the electron interaction effects on the NMR.

The conductivity has been measured on a bulk sample of n-type GaAs, with an electron concentration $n = 2.9 \times 10^{16}$ cm^{-3} close to the metal–insulator transition (MIT) ($k_F l_0 = 3.34 \approx \pi$) and a compensation ratio $K = 0.65$. The experiments have been performed down to 0.3 K and in magnetic fields up to 5.8 T, but our analysis does not exceed 1.5 T in order to avoid the influence of the Shubnikov–de Haas effect and of the normal positive magnetoresistance.

As the sample is very close to the MIT, the temperature behaviour of the conductivity consists of a $T^{1/3}$ regime below 1 K and the diffusion constant has the following form:

$$D = 2.549 \times 10^{-4} T^{1/3} + 8.5 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}. \tag{1}$$

The elastic mean free path l_0 has been evaluated to be 351 Å from the extrapolated value of $\sigma(T = 0)$.

2. Quantum corrections to conductivity in disordered semiconductors: theoretical background

The scope of this study is the weak localization occurring on the metallic side of the MIT. A qualitative explanation for the weak localization was given by Bergmann (1983), who considered the scattering of an electron along a closed loop following two opposite directions. Since the electrons are wavelike, in the low-temperature limit (the mean free path l_0 must be shorter than the coherence length L_φ) the waves experience multiple elastic scatterings by the impurities. Accumulating the same phase in both directions, they interfere constructively, causing the localization effect. When a magnetic field is applied, the interference is partially destroyed, bringing about a decrease in the resistance.

This NMR appears at temperature sufficiently low that the elastic scattering time τ_0 is much less than the inelastic one τ_ε. In materials where spin–orbit coupling is absent, τ_ε is generally likened to the phase coherence lifetime τ_φ.

The magnetoconductivity (MC) of a tridimensional disordered material without spin–orbit coupling has been expressed by Kawabata (1980) in this regime

$$\Delta\sigma(B, T) = \frac{e^2}{2\pi^2\hbar}\sqrt{\frac{eB}{\hbar}}\, f_3(\delta) \tag{2}$$

where

$$\delta = \frac{\hbar}{4DeB\tau_\varepsilon}$$

and

$$f_3(\delta) = \sum_{N=0}^{\infty}\left[2\left(\sqrt{N+1+\delta} - \sqrt{N+\delta}\right) - \frac{1}{\sqrt{N+\frac{1}{2}+\delta}}\right].$$

The relation (2) is valid under the conditions

$$\frac{\hbar}{m^* v_F \tau_0} \ll 1 \qquad \frac{eB\tau_0}{m^*} \ll 1 \qquad \sqrt{\frac{eB}{\hbar}}\, l_0 \ll 1 \tag{3}$$

namely $B < 0.5$ T in our case. Here, v_F indicates the Fermi speed of electrons, equal to $\hbar k_F/m^*$, where k_F is the Fermi wave vector.

Other quantum corrections to the conductivity arise from the enhancement of electron–electron interactions by the magnetic field. These effects are generally treated like perturbations within the scope of weak localization. Through a process similar to the weak localization, the diffusive nature of the electron motion is responsible for the enhancement of the electron–electron interactions.

These increases are classified into two different channels.

(i) The Cooper channel, also called the particle–particle channel, describes the interaction between electrons of antiparallel momenta. This effect produces a positive magnetoresistance through a dephasing process analogous to that discussed above.

(ii) The diffusion channel, or particle–hole channel, describes the interaction between electrons of parallel momenta. There is no magnetoresistance arising from the orbital interference in this channel. Nevertheless, the magnetic spin splitting leads to a positive magnetoresistance (Lee and Ramakrishnan 1982).

3. Electron interaction: orbital contribution

Altshuler *et al* (1981) have suggested an expression of the MC due to the density of states corrections caused by the Cooper channel interaction in the classically weak-field region. This effect is presumed to occur at weaker field than the Zeeman effect (Lee and Ramakrishnan 1982). In a one-valley semiconductor without spin–orbit coupling or superconducting behaviour, it gives

$$\Delta\sigma_A(B,\,T) = -\alpha\frac{e^2}{2\pi^2\hbar}g(B,\,T)\sqrt{\frac{eB}{\hbar}}\varphi_3\left(\frac{2DeB}{\pi kT}\right) \tag{4}$$

where $\alpha = 1$ in the case of GaAs and $g(B,\,T)$ is the renormalized coupling constant, expressed by Lee and Ramakrishnan (1982) as

$$g(B,\,T) = \frac{\bar{\lambda}}{1+\bar{\lambda}\ln(E_F/E_0)} \tag{5}$$

with $\bar{\lambda}$ being the effective electron–electron interaction constant. According to Altshuler *et al* (1981), E_0 has the following form:

$$E_0 = \max(DeB;\,1.764\;kT). \tag{6}$$

The function in φ_3 is defined for long phase coherence times by

$$\varphi_3(x) = \sqrt{\frac{\pi}{2x}}\int_0^\infty \frac{\sqrt{t}}{\sinh^2 t}\left(1-\frac{xt}{\sinh(xt)}\right)\,\mathrm{d}t. \tag{7}$$

Another form of the renormalized coupling constant $g(B,\,T)$ has been calculated by McLean and Tsuzuki (1984) who give

$$g^{-1}(B,\,T) = \ln\left(\frac{T_0}{T}\right) + \Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2}+\frac{DeB}{2\pi kT}\right) \tag{8}$$

where $\Psi(x)$ is the digamma function.

According to Howson and Gallagher (1988), the temperature T_0 in (8) is the superconducting transition temperature T_c in the case of superconducting materials, but for a normal metal it is the Fermi temperature $T_F = E_F/k$.

The orbital part of the positive magnetoresistance due to interactions has been added to the weak-localization contribution (2) with the intention of describing the experimental MC and extracting the inelastic scattering time at each temperature.

At first, we used the original form of Altshuler *et al* (1981), namely, our data have been analysed using equations (2), (4) and (5).

The result is obtained by a non-linear least-squares fit procedure where τ_ε and $\bar{\lambda}$ are the adjustable parameters and other constants are independently measurable. The chosen algorithm, due to Marquardt (1963), increases considerably the speed and the preciseness of the usual analytic least-squares fit method.

Subsequently, we have used a similar procedure with the coupling constant $g(B, T)$ given by McLean and Tsuzuki (equation (8)).

In equations (2) and (4), some functions have complicated analytic forms and cannot be avoided because of their physical meaning. It is then advisable to replace them by spline approximations. The functions $f_3(\delta)$, for example, is a sum of terms from zero to infinity for which we substitute a polynomial proposed by Baxter *et al* (1989).

The function $\varphi_3(x)$, that consists of an integral from zero to infinity, will also be represented by an Euler–MacLaurin series. A set of formulae for this function is given by Ousset *et al* (1985) and corrected by Baxter *et al* (1989). The corresponding approximation is written in the appendix with an accuracy of 2.5×10^{-4}.

The digamma functions that appear in equation (8) are also an infinite sum of terms and have been calculated by Oosset *et al* (1985) with an accuracy of 7.3×10^{-5} (See the appendix).

All these approximations have been used in the numerical program each time the functions or their derivatives were required. In order to reduce the number of fitting parameters and to improve the stability of the convergence, the Fermi energy E_F has been evaluated as $(\hbar(3\pi^2 n)^{2/3})/2m^*$ and the diffusion constant is given by equation (1).

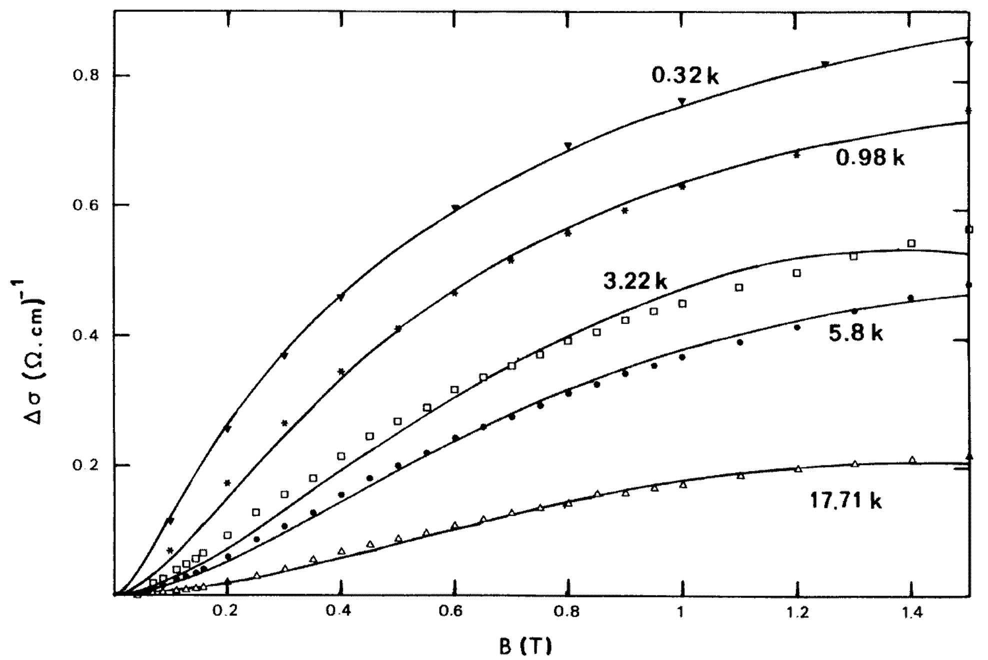
A first set of τ_ε values is provided by fitting the experimental MC to equations (2) and (4), including the form (5) of the coupling constant. It is shown in figure 1(*a*) that the agreement in the range of field $0 < B < 1.5$ T is acceptable for $T > 3$ K and excellent at very low temperature. However, it should be pointed out here that the results must be cautiously interpreted for several reasons. Indeed, the conditions of validity of some approximations are not fully satisfied. For example, the weak-localization model normally applies only if $B < 0.5$ T in our case. Secondly, the additivity of the individual contributions to the MC is to put into question in a system where $k_F l_0$ lies near π. The lack of a complete theory of weak localization close to the MIT leads us to assume the present one correct. Moreover, theoretical considerations (Altshuler and Aronov 1985, Morgan *et al* 1985) suggest that the interaction corrections may not be restricted to weak disorder ($k_F l_0 \gg 1$).

The obtained inelastic scattering time τ_ε is plotted against temperature in figure 2(*a*). The values have been compared to the model of Isawa (1984), who predicts that $\tau_\varepsilon \propto T^{-1}$ for $T < 6$ K.

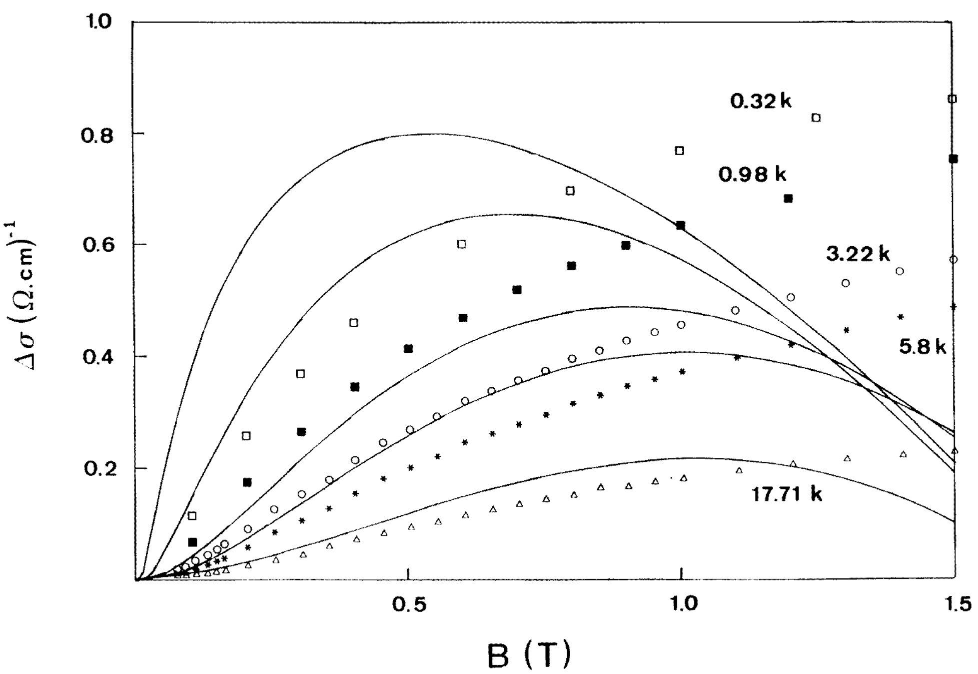
The experimental behaviour of τ_ε is actually found to be linear in T^{-1}, as well described by

$$\tau_\varepsilon = 8.946 \times 10^{-12} T^{-0.95}. \tag{9}$$

The electron–electron interaction constant $\bar{\lambda}$ has a particular variation with the temperature (see figure 3). It seems to stay in the neighbourhood of 2.1 in the high-temperature limit. Then it diminishes with T and tends to its second value $\bar{\lambda} = 0.46$. Considering that the fits improve in proportion as the temperature decreases, the low-temperature parameter must be the relevant one. Furthermore, whereas no significant explanation has been provided for the high-temperature value, the low-temperature value may be interpreted as equal to $F/2$ (Isawa and Fukuyama 1984), where the Hartree constant

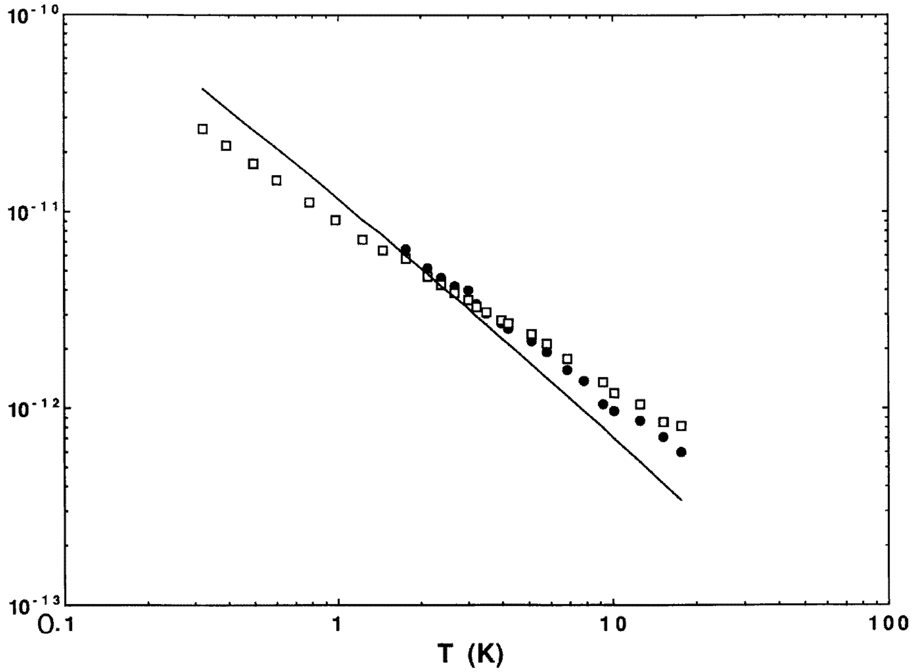


(*a*)

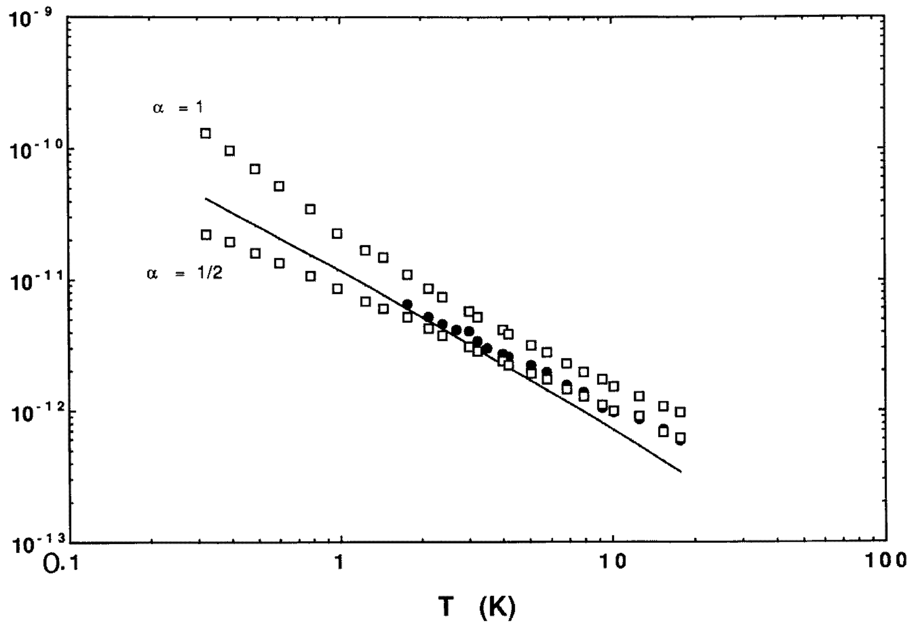


(*b*)

Figure 1. The variation of the magnetoconductivity with the field a and fit to the weak-localization and orbital electron interaction contributions (*a*) using equations (4) and (5) with $\alpha = 1$ or equations (4) and (8) with $\alpha = \frac{1}{2}$ and (*b*) using equations (4) and (8) with $\alpha = 1$.



(*a*)



(*b*)

Figure 2. The variation of the inelastic scattering time with temperature. A comparison between the weak localization model (●), the theoretical model of Isawa (——) and the model including orbital interaction (□) (*a*) using equations (4) and (5) and (*b*) using equations (4) and (8).

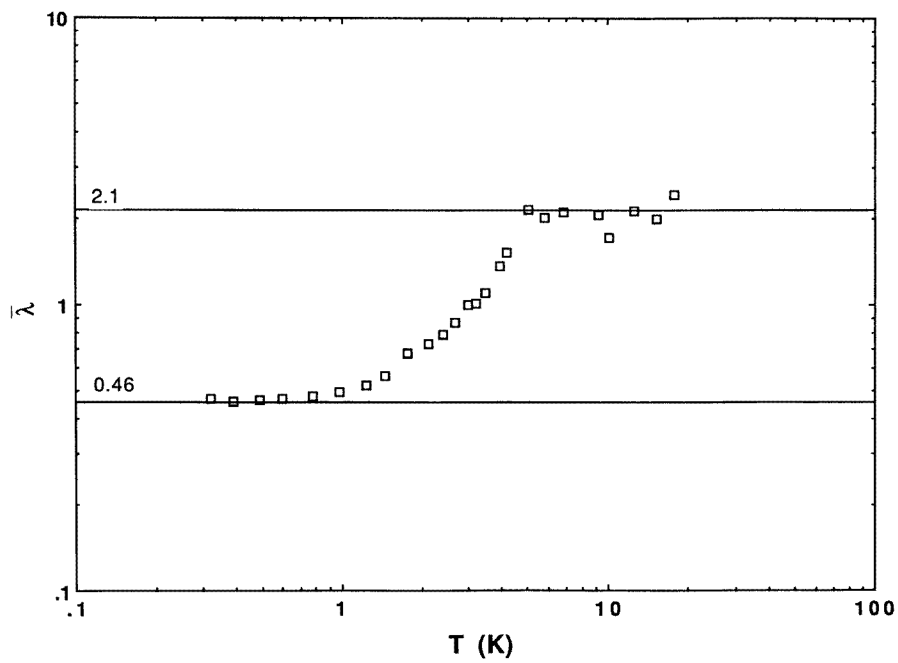


Figure 3. The variation of the interaction constant with temperature between its two stages in the high- and low-temperature limits.

F has been calculated in the Kleinman–Langreth screening approximation ($F_{KL} = 0.9$). The parameter F indicates the average of the screened Coulomb potential over the momenta q of the Fermi surface. This constant, contained between zero and unity, may have different values from the one calculated using the Thomas–Fermi approximation ($F_{TF} \approx 0.46$). In particular the Kleinman–Langreth approximation, which takes into account the correlation effects (Kleinman 1967, Langreth 1969), gives

$$F_{KL} = \frac{\ln(1+a)}{a} \qquad a = \left(\frac{K_s}{2k_F}\right)^2. \tag{10}$$

K_s is the screening parameter, determined by

$$K_s = k_F \sqrt{\left(1 - 0.158\left(\frac{k_{TF}}{2k_F}\right)^2\right)\left(1 + 0.158\left(\frac{k_{TF}}{2k_F}\right)^2\right)^{-1}}. \tag{11}$$

Knowing the Fermi vector k_F and the Thomas–Fermi parameter k_{TF}, we calculate $K_s = 9.03 \times 10^7\ \mathrm{m}^{-1}$ and $F_{KL} \approx 0.9$.

Subsequently, we have tried to fit the data to the same terms (2) and (4), this time using the renormalized coupling constant $g(B, T)$ given by equation (8), where the digamma functions are replaced by a development recommended by Ousset *et al* (1985). In such a case, the inelastic scattering time τ_ε remains the only fitting parameter and we expect to retrieve the values obtained with the first method.

Figure 1(*b*) shows that poor fits are obtained from this analysis if we consider $\alpha = 1$ in equation (4). Furthermore, the corresponding inelastic scattering time deviates strongly from its theoretical dependence (figure 2(*b*)). Experimentally, it gives

$$\tau_\varepsilon = 3.3 \times 10^{-11} T^{-1.22}. \tag{12}$$

Although this result is not too far from a T^{-1} behaviour, the discrepancies between these values, the model of Isawa at low temperature and the Kawabata experimental values at high temperature let us assume that the form (5) of $g(B, T)$ should be a better solution.

In fact, the previous dependence can be restored by putting $\alpha = \frac{1}{2}$ in equation (4), in which case relatively good fits may be performed again as in figure 1(a). Note that this prefactor of one-half is also found when calculated as a fitting parameter. According to Alschuler *et al* (1981), this coefficient, equal to unity in normal metals, may take the value of one-quarter when strong spin–orbit coupling is present. Since we have never heard about the presence of spin–orbit coupling in n-GaAs, we consider this result $\alpha = \frac{1}{2}$ as inconsistent with the observation of a strong NMR. However, since the model shows a good agreement with experience, we will assume that this contradiction is due to a failure in the theory.

We now take into account the diffusion channel in the description of the MC and discuss the relative importance of its contribution.

4. Electron interaction: Zeeman contribution

The splitting of the spin states by the magnetic field in the diffusion channel has been considered by Lee and Ramakrishnan (1982). According to them, the Hartree interaction between opposite spins is alone sensitive to the field. The exchange term implies a correlation between electrons of identical spin and remains unchanged by the magnetic field. The correction to the MC is then

$$\Delta\sigma_{LR}(B, T) = -\frac{e^2}{\hbar}\frac{F}{4\pi^2}\sqrt{\frac{kT}{2D\hbar}}g_3(h) \tag{13}$$

with $h = g\mu_B B/kT$, g being the Landé factor and μ_B the Bohr magneton. The function g_3 has the following analytic form:

$$g_3(h) = \int_0^\infty \mathrm{d}\Omega \frac{\mathrm{d}^2[\Omega N(\Omega)]}{\mathrm{d}^2\Omega}(\sqrt{\Omega + h} + \sqrt{|\Omega - h|} - 2\sqrt{\Omega}) \tag{14}$$

where $N(\Omega) = 1/(\mathrm{e}^\Omega - 1)$.

The contribution $\Delta\sigma_{LR}$ has been used in addition to equation (2) and (4) in order to describe the data up to 1.5 T. Here $g_3(h)$ has been replaced by the spline approximation of Ousset *et al* (1985) (see the appendix). The theoretical MC is then the sum of three terms (2), (4) and (13):

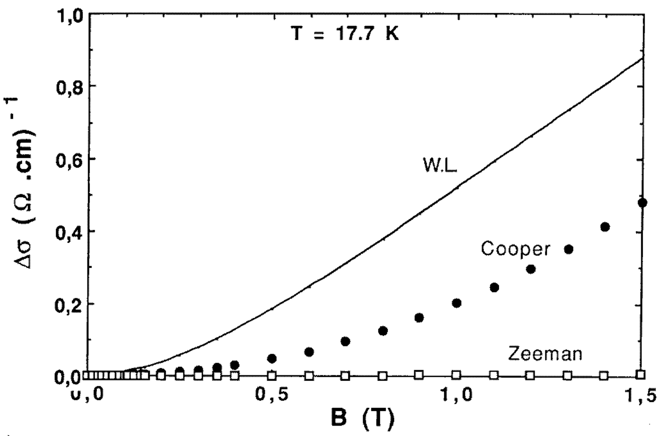
$$\Delta\sigma = \Delta\sigma_{WL} + \Delta\sigma_A + \Delta\sigma_{LR} \tag{15}$$

also written

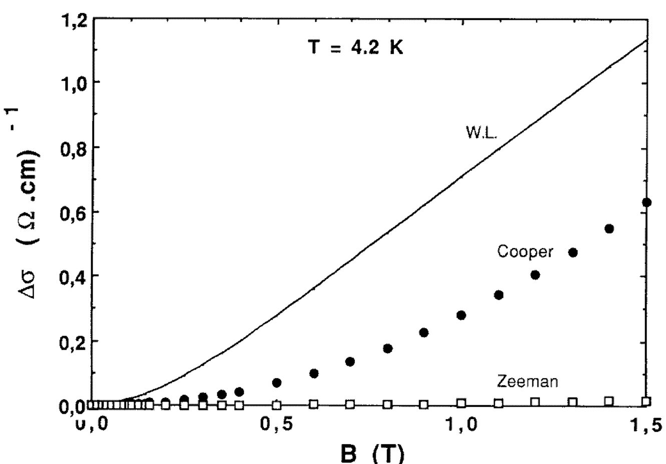
$$\Delta\sigma = 4.8\sqrt{B}\left[f_3(\delta) - g(B, T)\varphi_3(x) - \frac{F}{2}\sqrt{\frac{kT}{2DeB}}g_3(h)\right] \tag{16}$$

with $\delta = \hbar/4DeB\tau_\varepsilon$, $x = 2DeB/\pi kT$ and $h = g\mu_B B/kT$.

We have tentatively fitted the experimental MC to this relation, firstly replacing $g(B, T)$ by the expression (5). The result is as correct as if we consider weak localization and the Cooper channel only (figure 1(a)). From this analysis we could evaluate the three adjustable parameters involved in equation (16). The values of τ_ε and $\bar{\lambda}$ are the same as previously found in section 3. However, the fitted values of F are surprisingly very important ($F \gg 1$) in the high-temperature limit and tend to zero at low temperature. This result has no physical meaning since F must be in principle a constant less than unity for a one-valley conduction band. Such a dispersion of the F values with the temperature may originate from the number of fitting parameters. Indeed, we noticed that the fit depended



(*a*)



(*b*)

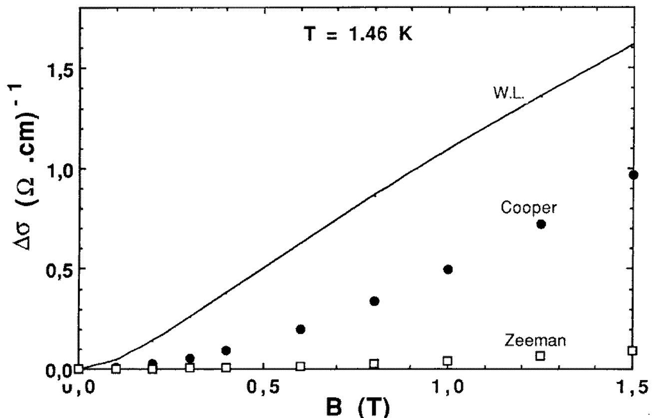
Figure 4. The magnitudes of the weak-localization, the orbital interaction and the Zeeman interaction contributions to the magnetoconductivity, calculated using fitted parameters and $F = 0.9$ in equation (16). Plots against the magnetic field (*a*) at 17.7 K, (*b*) at 4.21 K, (*c*) at 1.46 K and (*d*) at 0.32 K.

on the initial conditions, showing that evolution of the chi-square with τ_ε, $\bar{\lambda}$ and f presents several valleys.

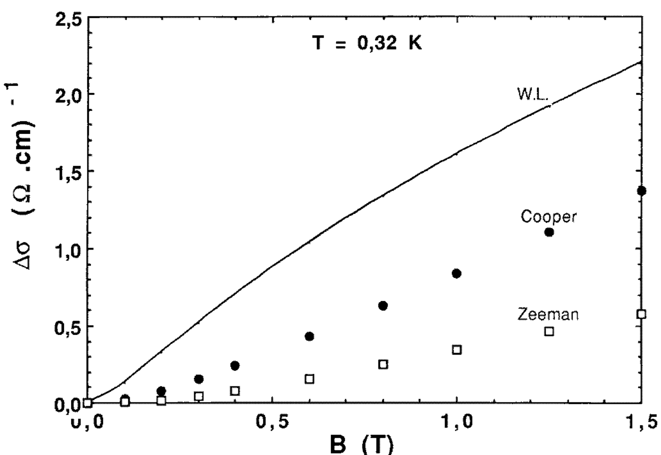
Another method to include the interaction effect due to the diffusion channel is to use the form of $g(B, T)$ given by equation (8) and to take $\alpha = \frac{1}{2}$ in equation (4), as we have already seen in section 3. This implies a two-parameter least-squares fit procedure, which appears rather more stable. In this case, the behaviour of the fitted MC is almost the same as in figure 1(*a*) and the experimental dependence of the inelastic scattering time is given by

$$\tau_\varepsilon = 8.18 \times 10^{-12} T^{-0.87} \tag{17}$$

which is very close to the behaviour obtained without the Zeeman contribution (figure 2(*b*)). Nevertheless, the Hartree constant F is found to remain greater than unity in the high-



(*c*)



(*d*)

Figure 4. (Continued)

temperature range and to tend towards zero when lowering the temperature. Thus the Zeeman effect seems to be negligible in GaAs at low temperature.

At this stage of the description, we have to discuss the significance of the parameter F, which normally should be a constant smaller than unity. In figure 4 are reported the contributions to the MC of the three terms appearing in equation (16), at four different temperatures. These terms are calculated using the fitted parameters, except for F, which is taken to be equal to 0.9. It can be observed that, above 1.5 K, the diffusion term has negligible influence on the total magnetoresistance. This might be related to the fact that $\mu_B B < kT$ in the whole range of magnetic field for $T > 1$ K. As a result, the fits are insensitive to the value of F in the high-temperature range and the obtained values of F are generally strong in order to compensate the smallness of this term in the total MC.

At low temperature, in contrast, the Zeeman term is meanful (figure 4(*d*)) but the fit to equations (16) and (8) leads to $F = 0$. Surprisingly, this value is contradictory to the one calculated before in the Kleinman–Langreth approximation ($F = 0.9$). In fact, this inconsistency can be cancelled if we include the attractive interaction via virtual phonons, as suggested by Howson and Gallagher (1988). In this case, the Hartree constant F must be replaced with

$$F^* = F - 2\lambda_p \tag{18}$$

where λ_p is the electron–phonon coupling constant. The fitted value of F would be in reality $F^* \approx 0$, suggesting that $\lambda_p \approx F/2$ in this sample.

5. Conclusion

In this paper, we have tested several models for the NMR in the weakly localized regime, taking into account the electron–electron interaction enhancement. The corresponding corrections include an orbital part and a term due to the Zeeman effect. We have shown that the Cooper channel contribution alone is enough to describe properly the experimental MC in this range of magentic field. In the low-temperature range, the parameters experimentally found using the different methods have coherent values and the inelastic scattering rate τ_ε^{-1} seems to be linear with T. This persistent dependence has been explained by Belitz and Wysokinski (1987) as a result of the Wegner scaling theory applied to the critical current dynamics.

According to Hikami *et al* (1980), the appearance of localized spins at the mobility edge leads to the saturation in the temperature dependence of τ_ε. However, such a behaviour has not been observed in our sample, even in the low-temperature limit. The impurity concentration may not be close enough to its critical value for this phenomenon to occur.

Moreover, there still remains some doubt concerning the meaning of the spin–orbit parameter α, introduced in the interaction term of Altshuler *et al* (1981). A comparative study of the different contributions involved in the MC has made it apparent that the Cooper channel is the dominant interaction effect in n-GaAs near the MIT. This result is opposed to several experimental works on more metallic systems (see for example Sahnoune *et al* 1992, Kyllesbech Larsen *et al* 1994). Furthermore, while it is well established that the Zeeman effect has no influence on the MC above 1 K, its apparent significance at low temperature seems neutralized by the smallness of its amplitude ($F^* = 0$).

Finally, it would be worthwhile to perform measurements at lower temperature (20 mK for example) in order to check the stability of the presumed behaviours.

Acknowledgment

The experiments at the lowest temperatures were carried out with A Briggs at the CRTBT (CNRS) in Grenoble. We are grateful to him for his assistance in this part of the work.

Appendix

Euler–MacLaurin developments of several functions used in the least-squares fit procedures are reported below.

(i) *Kawabata's function $f_e(\delta)$ with an accuracy better than 0.1%*:

$$f_3(\delta) = 2(\sqrt{2+\delta} - \sqrt{\delta}) - \left[\left(\frac{1}{2}+\delta\right)^{-1/2} + \left(\frac{3}{2}+\delta\right)^{-1/2}\right] + \frac{1}{48}(2.03+\delta)^{-3/2}.$$

(ii) *Function $\varphi_3(x)$ of Alshuler et al with an accuracy of* 2.5×10^{-4}:
for $x \leqslant 0.7$

$$\varphi_3(x) = 0.329\,25x^{3/2} - 0.118\,94x^{7/2} + 0.107\,53x^{11/2} - 0.0636x^{6.63}$$

for $0.7 \leqslant x \leqslant 2.4$

$$\varphi_3(x) = -0.030\,43 + 0.226\,16x + 0.141\,04x^2 - 0.102\,93x^3 + 0.027\,59x^4 - 0.0028x^5$$

for $x \geqslant 2.4$

$$\varphi_3(x) = 1.900\,344 - \frac{2.293\,92}{\sqrt{x}} + \frac{1.2266}{x^2} - \frac{0.826}{x^{7/2}}.$$

(iii) *Function $g_3(h)$ of Lee and Ramakrishnan with an accuracy of* 2.5×10^{-4}:
for $h \leqslant 3$

$$g_3(h) = 5.6464 \times 10^{-2}h^2 - 1.4759 \times 10^{-3}h^4 + 4.2747 \times 10^{-5}h^6 - 1.5351 \times 10^{-6}h^8 + 6 \times 10^{-8}h^{10}$$

for $3 \leqslant h \leqslant 8$

$$g_3(h) = 0.645\,48 + 0.235(h-4) - 7.45 \times 10^{-4}(h-4)^2 - 2.94 \times 10^{-3}(h-4)^3 + 6.32 \times 10^{-4}(h-4)^4 - 5.22 \times 10^{-5}(h-4)^5$$

for $h \geqslant 8$

$$g_3(h) = \sqrt{h} - 1.2942 - \frac{\pi^2}{12h^{3/2}} - \frac{\pi^4}{16h^{7/2}} - \frac{5\pi^6}{32h^{11/2}}.$$

(iv) *Digamma functions in the expression of McLean and Tsuzuki, with an accuracy of* 7.3×10^{-5}:

$$\Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2} + x\right) = -4x\left\{\frac{1}{1+2x} + \frac{1}{9+6x} + \frac{85+32x}{150(5+2x)^2}\right\} - \ln\left(1 + \frac{2x}{5}\right).$$

References

Altshuler B L and Aronov A G 1985 *Electron–Electron Interactions in Disordered Systems* vol 1, ed A L Efros and M Pollak (Amsterdam: North-Holland)
Altshuler B L, Aronov A G, Larkin A I and Khmelnitskii D E 1981 *Sov. Phys.–JETP* **54** 411
Baxter D V, Richter R, Trudeau M L, Cochrane R W and Strom-Olsen J O 1989 *J. Physique* **50** 1673
Belitz D and Wysokinski K I 1987 *Phys. Rev.* B **36** 9333
Bergmann G 1983 *Phys. Rev.* B **28** 2917
Bogdanovich S, Dai P, Sarachik M P and Dobrosavljevic V 1995 *Phys. Rev. Lett.* **74** 2543
Capoen B, Biskupski G and Briggs A 1993 *J. Phys.: Condens. Matter* **5** 2545
Dynes R C, Geballe T H, Hull G W and Garno J P 1983 *Phys. Rev.* B **27** 5188
Fritzsche H and Lark-Horovitz K 1955 *Phys. Rev.* **99** 400
Hikami S, Larkin A I and Nagaoka Y 1980 *Prog. Theor. Phys. Lett.* **63** 707
Howson M A and Gallagher B L 1988 *Phys. Rep.* **170** 265
Isawa Y 1984 *J. Phys. Soc. Japan* **53** 2865
Isawa Y and Fukuyama H 1984 *J. Phys. Soc. Japan* **53** 1415
Kawabata A 1980 *Solid State Commun.* **34** 431
Kleinman L 1967 *Phys. Rev.* **160** 585
Kyllesbech Larsen K, Van Hove M, Lauwers A, Donaton R A, Maex K and Van Rossum M 1994 *Phys. Rev.* B **50** 14 200
Langreth D 1969 *Phys. Rev.* **181** 753
Lee P A and Ramakrishnan T V 1982 *Phys. Rev.* B **26** 4009
Marquardt D W 1963 *J. Soc. Ind. Appl. Math.* **11** 431
McLean W L and Tsuzuki T 1984 *Phys. Rev.* B **29** 503
Morgan G J, Howson M A and Saub K 1985 *J. Physique* **13** 2157
Morita S, Mikoshiba N, Koike Y, Fukase T, Kitigawa M and Ishida S 1984 *J. Phys. Soc. Japan* **53** 40
Ootuka Y, Matsuoka H and Kobayashi S 1987 *Disordered Semiconductors* ed M A Kastner *et al* (New York: Plenum)
Ousset J C, Askenazy S, Rakoto H and Broto J M 1985 *J. Physique* **46** 2145
Sahnoune A, Ström-Olsen J O and Zaluska A 1992 *Phys. Rev.* B **46** 10 629